Generalized entanglement measure for continuous-variable systems

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Concurrence, introduced by Hill and Wootters [Hill and Wootters, Phys. Rev. Lett. **78**, 5022 (1997)], provides an important measure of entanglement for a general pair of qubits that is strictly positive for entangled states and vanishes for all separable states. We present an extension of the entanglement measure to general pure continuous variable states of multiple degrees of freedom by generalizing the Lagrange's identity and wedge product framework proposed by Bhaskara and Panigrahi [Bhaskara and Panigrahi, Quantum Inf. Process. **16**, 118 (2017)] for pure discrete-variable systems in arbitrary dimensions and extending the concept to mixed continuous-variable states. A family of faithful entanglement measures is constructed that admits necessary and sufficient conditions for separability across arbitrary bipartitions presented by Vedral *et al.* [Vedral, Plenio, Rippin, and Knight, Phys. Rev. Lett. **78**, 2275 (1997)]. The computed entanglement measure in the present approach for general Gaussian states, pair-coherent states, and non-Gaussian continuous-variable Bell states matches with known results. We also quantify entanglement of phase-randomized squeezed states and superposition of squeezed states. Our results also simplify several results in quantum entanglement theory.

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I. INTRODUCTION

Quantum entanglement, having played a fundamental role in quantum information theory, is also finding its context in deeper questions, including on the origin of space-time [1], quantum field theories [2], many-body physics [3], Berry phase [4], and quantum gravity [5]. Detecting the presence of such a resource and quantifying it faithfully for the general case of continuous-variable systems would have far-reaching applications beyond quantum computation.

Previous works, including the extensions of Peres-Horodecki criteria by Simon [6], Agarwal [7], and Werner [8]; nonlinear maps on matrices by Giedke *et al.* [9]; and criteria based on uncertainty principles by Duan *et al.* [10] and Hillery, Nha, and Zubairy [11,12], for continuous-variable (CV) systems have provided the necessary conditions for the class of non-Gaussian states for manifesting quantum optics. Note, however, various geometry-based approaches exist to quantify entanglement [13–18]. In this paper, we provide a family of faithful measures of entanglement, as an extension to concurrence [14], admitting necessary and sufficient criteria for measurement of entanglement [19] across arbitrary bipartitions and degrees of freedom for general pure and mixed CV states, identifying an inherent geometry of entanglement in a similar spirit with examples of general Gaussian states, phase-matched squeezed states, pair-coherent states, superposition of squeezed states, and non-Gaussian CV Bell states. We comment on the connections to the widely used measures in the concluding section.

II. PRELIMINARIES

We introduce the notion of a genuine entanglement measure based on the wedge product and the Lagrange-Brahmagupta identity for discrete-variable systems. In an n-dimensional complex space, the vectors $\vec{p}$ and $\vec{q}$ can be written as $\vec{p} = \sum_i p_i e_i$ and $\vec{q} = \sum_j q_j e_j$, respectively. The bivector $\vec{p} \wedge \vec{q}$ represents an oriented parallelepiped with sides of vectors $\vec{p}$ and $\vec{q}$:

$$\vec{p} \wedge \vec{q} = \sum_{i<j} (p_i q_j - p_j q_i) e_i \wedge e_j.$$

The Lagrange-Brahmagupta identity takes the form

$$\| \vec{a} \|^2 \| \vec{b} \|^2 - |\vec{a} \cdot \vec{b}|^2 = \| \vec{a} \wedge \vec{b} \|^2$$

for vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{C}^m$. Without loss of generality, one can take $d_A = \dim(\mathcal{H}_A) \leqslant \dim(\mathcal{H}_B) = d_B$. If $\{|\phi_i'\rangle, i = 1, 2, ., d_A\}$ is an orthonormal basis of $\mathcal{H}_A$, it follows that

$$|\Psi'\rangle = \sum_{i=0}^{d_A} |\phi_i'\rangle\langle\phi_i'|\Psi'\rangle.$$

Entanglement measure of the bipartition $A|B$ defined in terms of wedge products [14] of postmeasurement vectors can be written as

$$C_{A|B}^2 = 4 \sum_{i<j} |\langle\phi_i'|\Psi'\rangle \wedge \langle\phi_j'|\Psi'\rangle|^2,$$

where i and j take values from 0 to d_A. The condition for separability across this bipartition is $C_{A|B}^2 = 0$.

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Maximal $C_{A|B}$ for a particular bipartition $A|B$ will correspond to the following conditions:

$$\langle\phi_i'|\Psi'\rangle^\dagger\langle\phi_j'|\Psi'\rangle = 0 \quad \forall \quad i \neq j,$$
$$|\langle\phi_i'|\Psi'\rangle| = |\langle\phi_j'|\Psi'\rangle| \quad \forall \quad i, j.$$

Two-qubit case. For a two-qubit system, the general state $|\psi\rangle$ in the computational basis is given by

$$|\psi_{AB}'\rangle = p|00\rangle + q|01\rangle + r|10\rangle + s|11\rangle,$$

where $|ij\rangle = |i_A\rangle \otimes |j_B\rangle$ and $a, b, c, d \in \mathbb{C}$, satisfying the normalization condition

$$|p|^2 + |q|^2 + |r|^2 + |s|^2 = 1.$$

The generalized concurrence measure in terms of the wedge product (as a measure of entanglement) for $|\ \psi'\rangle_{AB}$ has been obtained earlier as [14,18]

$$\mathcal{E} = 2|\langle 0_A \mid \psi'\rangle \wedge \langle 1_A|\psi'\rangle|.$$

These two conditions lead to the general form of maximally entangled states for a two-qubit system:

$$\bar{a}c + \bar{b}d = 0,$$
$$|a|^2 + |b|^2 = |c|^2 + |d|^2.$$

We can also get the Bell states

$$|\psi_\pm'\rangle = \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}},$$
$$|\phi_\pm'\rangle = \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}}.$$

Three-qubit case. The general state is given by

$$|\psi_{ABC}'\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle,$$

where $a_i \in C$, $i = 1$–7, satisfies the normalization condition $\sum_{i=0}^{7} |a_i|^2 = 1$. The measure of entanglement $\mathcal{E}$ is given by the sum of concurrence corresponding to all three bipartitions [14]:

$$\mathcal{E} = \mathcal{E}_{A|BC} + \mathcal{E}_{B|AC} + \mathcal{E}_{C|AB}.$$

In the wedge product formalism, we can write the concurrence as

$$\mathcal{E} = 2\sum_{i=A,B,C} |\langle 0_i|\psi_{ABC}'\rangle \wedge \langle 1_i|\psi_{ABC}'\rangle|.$$

$|\Psi_{ABC}'\rangle$ can be written in the following form:

$$|\Psi_{ABC}'\rangle = |0_A\rangle\langle 0_A|\Psi_{ABC}'\rangle + |1_A\rangle\langle 1_A|\Psi_{ABC}'\rangle.$$

The conditions for maximally entangled states are as follows: $\langle 0_i|\psi_{ABC}'\rangle$ must be orthogonal to $\langle 1_i|\psi_{ABC}'\rangle$ and $|\langle 0_i|\psi_{ABC}'\rangle| = |\langle 1_i|\psi_{ABC}'\rangle|$ for each $i = A$, B, and C. This maximally entangled state (Greenberger-Horne-Zeilinger state) by using the above conditions is defined as

$$|\psi'\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

We then explicitly explain in the next section a much more refined and efficient form of generalized entanglement measure (GEM) for continuous-variable systems that is based on the extended wedge product and the Lagrange-Brahmagupta identity. The main advantage of our proposed GEM is that less computational effort is required for its evaluation.

III. GENERALIZED ENTANGLEMENT MEASURE (GEM) FOR PURE CV STATES

We define separability for pure states in the context of CV systems for future convenience. Consider an n-degree-of-freedom quantum system. Let $P|Q$ be a bipartition across the degrees of freedom of this composite (whole) system $P \cup Q$, with respective infinite-dimensional Hilbert spaces $\mathcal{H}_P$ and $\mathcal{H}_Q$ for the states of the subsystems P and Q, and then the state space of the composite system is given by the tensor product $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_Q$. If a pure state $|\psi\rangle \in \mathcal{H}$ of the composite system with $\rho_\psi = |\psi\rangle\langle\psi|$ can be written in the form

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle, \qquad \text{i.e., } \rho_\psi = \rho_\phi \otimes \rho_\chi,$$

where $|\phi\rangle \in \mathcal{H}_P$ and $|\chi\rangle \in \mathcal{H}_Q$ are the pure states of the subsystems P and Q, respectively, with $\rho_\phi = |\phi\rangle\langle\phi|$ and $\rho_\chi = |\chi\rangle\langle\chi|$, then the system is said to be separable across the bipartition $P|Q$. Otherwise, the subsystems P and Q are said to be entangled.

Consider a general n-degree-of-freedom pure CV state $|\psi\rangle$ with the degrees of freedom taking continuous values and labeled by $\{x_1, x_2, \ldots, x_n\}$ in an orthonormal basis with $\langle\vec{x}'|\vec{x}\rangle = \delta(\vec{x} - \vec{x}')$ and $\int |\vec{x}\rangle\langle\vec{x}|\, d\vec{x} = 1$ as

$$|\psi\rangle = \int \phi(x_1, \ldots, x_n)|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle d\vec{x}, \tag{1}$$

with $\langle\psi|\psi\rangle = 1$, i.e.,

$$\int \phi^*(x_1, \ldots, x_n)\phi(x_1, \ldots, x_n)d\vec{x} = 1,$$

where $\vec{x} = (x_1, x_2, \ldots, x_n)$, $d\vec{x} \equiv d^n x = dx_1 dx_2 \ldots dx_n$, and δ is the Dirac delta function of appropriate dimension. Note that the limits of the integrals are over the appropriate continuous range of values for the degrees of freedom (commonly, $-\infty$ to $+\infty$) unless otherwise specified. By an n-degree-of-freedom system one could mean, for instance, a system of n particles in one spatial dimension, a system of k particles in three dimensions where $n = 3k$, or a quantum optics system having multiple modes. The physical state $|\psi\rangle$ exists in an infinite-dimensional Hilbert space spanned by $\{|x\rangle\}$. Note that, unlike the case of discrete-variable (DV) systems, the basis states $\{|x\rangle\}$ by themselves are not normalizable and hence are nonphysical.

The generalized entanglement measure (GEM) for pure CV states will now be defined as

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \iint \left| \int \phi(y_1', y_2', \ldots, y_m', x_{m+1}, \ldots, x_n)\phi^*(y_1, y_2, \ldots, y_m, x_{m+1}, \ldots, x_n)d^{n-m}x \right|^2 d^m y d^m y'\right]. \tag{2}$$

We can define

$$\tilde{\Phi}(\mathcal{X}) = \langle \mathcal{X}|\mathbb{N}_1^T|\psi\rangle\langle \mathcal{X}|\mathbb{N}_2^T|\psi\rangle = \phi(\mathbb{N}_1\mathcal{X})\phi(\mathbb{N}_2\mathcal{X}), \tag{3}$$

$\mathbb{N}_1 = [1_{n\times n} \quad 0_{n\times n}]_{n\times 2n}$, $\mathbb{N}_1 = [0_{n\times n} \quad 1_{n\times n}]_{n\times 2n}$,

$$\mathbb{M} = \begin{bmatrix} 1_{m\times m} & \\ & 0_{(n-m)\times(n-m)} \end{bmatrix}_{2n\times 2n}, \quad \wedge_m = \begin{bmatrix} 1_{n\times n} - \mathbb{M}_{n\times n} & \mathbb{M}_{n\times n} \\ \mathbb{M}_{n\times n} & 1_{n\times n} - \mathbb{M}_{n\times n} \end{bmatrix}_{2n\times 2n}.$$

Equation (5) can be written in terms of ϕ and $\wedge_m$ as

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \mathrm{Re}\int \tilde{\Phi}(\mathcal{X})\tilde{\Phi}^*(\wedge_m\mathcal{X})d^{2n}\mathcal{X}\right]. \tag{4}$$

Proposition. The state ψ is said to be separable across the bipartition $\mathcal{M}|\overline{\mathcal{M}}$ if and only if ψ is expressible as

$$\begin{aligned}|\psi\rangle &= \int \phi(x_1, x_2, \dots, x_n)|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle d\vec{x} \\ &= \left[\int \phi_{\mathcal{M}}(x_1, \dots, x_m)|x_1 \dots x_m\rangle d^m x\right] \otimes \left[\int \phi_{\overline{\mathcal{M}}}(x_{m+1}, \dots, x_n)|x_{m+1} \dots x_n\rangle d^{n-m} x\right].\end{aligned} \tag{5}$$

Proof. Consider the bipartite separability of a particular set $\mathcal{M}$ of m degrees ($m < n$) out of the n degrees of freedom of the system. Without any loss of generality, let the m degrees be labeled by $\{1, 2, \dots, m\}$, so that the degrees labeled by $\{m+1, m+2, \dots, n\}$ represent the rest of the $(n-m)$ degrees of freedom belonging to the complement set $\overline{\mathcal{M}}$. The state $|\psi\rangle$ is said to be separable across the bipartition $\mathcal{M}|\overline{\mathcal{M}}$ if and only if $|\psi\rangle$ is expressible as

$$\left[\int \phi_{\mathcal{M}}(x_1, \dots, x_m)|x_1 \dots x_m\rangle d^m x\right] \otimes \left[\int \phi_{\overline{\mathcal{M}}}(x_{m+1}, \dots, x_n)|x_{m+1} \dots x_n\rangle d^{n-m} x\right],$$

where $\phi_{\mathcal{M}}$ is the normalized pure state of the subsystem $\mathcal{M}$, $|x_1 \dots x_m\rangle \equiv |x_1\rangle \otimes \cdots \otimes |x_m\rangle$, $d^m x \equiv dx_1 dx_2 \dots dx_m$, and similarly $\phi_{\overline{\mathcal{M}}}$ is the normalized pure state of the subsystem $\overline{\mathcal{M}}$, $|x_{m+1} \dots x_n\rangle \equiv |x_{m+1}\rangle \otimes \cdots \otimes |x_n\rangle$, and $d^{n-m}x \equiv dx_{m+1}dx_{m+2} \dots dx_n$.

One may rewrite the state $|\psi\rangle$, defined in Eq. (1), as

$$\int \left[|x_1 \dots x_m\rangle \otimes \left(\int \phi(x_1, \dots, x_n)|x_{m+1} \dots x_n\rangle d^{n-m}x\right) d^m x\right].$$

By noting that

$$\langle x_1' x_2' \dots x_m'|\psi\rangle = \int \phi(x_1, \dots, x_n) \underbrace{\langle x_1' x_2' \dots x_m'|x_1 x_2 \dots x_n\rangle}_{\delta(x_1 - x_1', \dots, x_m - x_m')|x_{m+1} \dots x_n\rangle} d^n x = \int \phi(x_1', x_2', \dots, x_m', x_{m+1}, \dots, x_n)|x_{m+1} \dots x_n\rangle d^{n-m}x,$$

one may express $|\psi\rangle$ as

$$|\psi\rangle = \int [|x_1 \dots x_m\rangle \otimes (\langle x_1 x_2 \dots x_m|\psi\rangle) d^m x]. \tag{6}$$

Observe that, for the separability of $|\psi\rangle$ across $\mathcal{M}|\overline{\mathcal{M}}$, each of the vectors $\langle x_1 x_2 \dots x_m|\psi\rangle$ in Eq. (6) must be mutually "parallel" for the m degree-of-freedom state to factor out; i.e., for each $\vec{r} = (r_1, r_2, \dots, r_m)$ one needs

$$\langle r_1 r_2 \dots r_m|\psi\rangle = c_{\vec{r}\vec{s}}\langle s_1 s_2 \dots s_m|\psi\rangle$$

for any $\vec{s} = (s_1, s_2, \dots, s_m)$, where $c_{\vec{r}\vec{s}}$ is some complex scalar, for separability. This becomes evident once one chooses, say, $(s_1, s_2, \dots, s_m) = (0, 0, \dots, 0) = \vec{0}$, so that

$$\langle r_1 r_2 \dots r_m|\psi\rangle = c(r_1, r_2, \dots, r_m)\langle \underbrace{00\dots0}_{m}|\psi\rangle,$$

where $c(r_1, r_2, \dots, r_m)$ is some complex scalar. Substituting this back in Eq. (6), one can see the state becomes separable (with constant k ensuring the normalization of each of the subsystem's states) as

$$|\psi\rangle = \int [|x_1 \dots x_m\rangle \otimes c(x_1, x_2, \dots, x_m)\langle 00\dots0|\psi\rangle d^m x] = \underbrace{\left(k\int |x_1 \dots x_m\rangle c(x_1, x_2, \dots, x_m) d^m x\right)}_{\text{state of } \mathcal{M}} \otimes \underbrace{\frac{1}{k}\langle 00\dots0|\psi\rangle}_{\text{state of } \overline{\mathcal{M}}},$$

$$|\psi\rangle = \left(\int \phi(x_1, \dots, x_m)|x_1 \dots x_m\rangle d^m x\right) \otimes (\phi(0, \dots, 0, x_{m+1}, \dots, x_n)|x_{m+1} \dots x_n\rangle d^{n-m}x). \tag{7}$$

Interestingly, therefore, one may note that, even if a single "pair" of elements of the continuous, infinite set of vectors $\{\langle x_1x_2\dots x_m|\psi\rangle\}$ over the continuous variables $x_1,\dots,x_m$ is not mutually parallel, this adds to the presence of entanglement. We express this condition for separability using the notion of a wedge product extended to multivariable complex-valued function spaces based on the framework proposed in Ref. [14] for general pure discrete-0variable systems in arbitrary dimensions.

Theorem. If a single pair of elements of the continuous, infinite set of vectors $\langle x_1\dots x_m|\psi\rangle$ over the continuous variables $x_1,\dots,x_m$ is not mutually parallel, the family of faithful measures of entanglement across the bipartition $\mathcal{M}|\overline{\mathcal{M}}$ is defined as

$$\mathcal{E}_{\mathcal{M}}^2=2\left[1-\iint\left|\int\phi(y_1',y_2',\dots,y_m',x_{m+1},\dots,x_n)\phi^*(y_1,y_2,\dots,y_m,x_{m+1},\dots,x_n)d^{n-m}x\right|^2d^myd^my'\right]. \tag{8}$$

Proof. In geometric algebra [20], the wedge product of two vectors is seen as a particular generalization of cross products to higher dimensions. We construct such a notion for the case of complex, infinite-dimensional vector spaces. Consider two vectors $\vec{a}$ and $\vec{b}$ in the complex, infinite-dimensional space as

$$\vec{a}=\int f(\vec{x})|\vec{x}\rangle d\vec{x},\quad \vec{b}=\int g(\vec{x})|\vec{x}\rangle d\vec{x}$$

in the continuous orthonormal basis set $\{|\vec{x}\rangle\}$, with $\langle\vec{x}'|\vec{x}\rangle=\delta(\vec{x}-\vec{x}')$ and $\int|\vec{x}\rangle\langle\vec{x}|d\vec{x}=1$, where $\vec{x}=(x_1,\dots,x_n)$. Then the wedge product of $\vec{a}$ and $\vec{b}$ in the interval $(\vec{t},\vec{u})$ is defined as a bivector in an "exterior" space with the continuous basis set $\{|\vec{x}\rangle\wedge|\vec{x}'\rangle\}_{x'>x}$, stipulating that $|\vec{x}\rangle\wedge|\vec{x}'\rangle=-|\vec{x}'\rangle\wedge|\vec{x}\rangle$ and $|\vec{x}\rangle\wedge|\vec{x}\rangle=0$, as

$$\vec{a}\wedge\vec{b}=\int_{\vec{x}=\vec{t}}^{\vec{u}}\int_{\vec{x}'=\vec{x}}^{\vec{u}}[f(\vec{x})g(\vec{x}')-f(\vec{x}')g(\vec{x})]|\vec{x}\rangle\wedge|\vec{x}'\rangle d\vec{x}'d\vec{x},$$

where $\vec{t}=(t_1,t_2,\dots,t_n)$ and $\vec{u}=(u_1,u_2,\dots,u_n)$. Therefore, one may note $\vec{a}\wedge\vec{b}=0\iff\vec{b}=k\vec{a}$, and $\vec{a}\wedge\vec{b}=-\vec{b}\wedge\vec{a}$, by definition, for some complex scalar k and vectors $\vec{a}$ and $\vec{b}$.

This notion of an extended wedge product allows one to write the separability condition in a compact and useful form. Since one requires that each of the vectors in the continuous set $\{\langle x_1x_2\dots x_m|\psi\rangle\}$ to be mutually "parallel" for the separability across $\mathcal{M}|\overline{\mathcal{M}}$, their mutual wedge products must vanish, equivalently, for separability. This is a necessary and sufficient condition for separability as noted before. Hence, one may construct a family of faithful measures of entanglement, parametrized by f, p, and q, across the bipartition as

$$\mathcal{E}_{\mathcal{M}}=\left[\iint f(||\langle y_1'y_2'\dots y_m'|\psi\rangle\wedge\langle y_1y_2\dots y_m|\psi\rangle||_{\mathrm{p}})d^myd^my'\right]^{1/q},$$

where $f:\mathbb{R}\to\mathbb{R}$ with $f(x)=0$ iff $x=0$ so that $\mathcal{E}_{\mathcal{M}}=0\iff$ separability, in addition to f being a monotonic and strictly increasing function in $\mathbb{R}^+$ and $q\in\mathbb{R}^+$ so that $\mathcal{E}_{\mathcal{M}}>0$ measures entanglement faithfully; the p-norm is computed in the basis $\{|x_{m+1}\dots x_n\rangle\wedge|x_{m+1}'\dots x_n'\rangle\}_{x'>x}$, and $\langle y_1'y_2'\dots y_m'|\psi\rangle\wedge\langle y_1y_2\dots y_m|\psi\rangle\equiv$

$$\int_{\vec{x}=-\infty}^{+\infty}\int_{\vec{x}'=\vec{x}}^{+\infty}[\phi(y_1',y_2',\dots,y_m',x_{m+1},\dots,x_n)\,\phi(y_1,y_2,\dots,y_m,x_{m+1}',\dots,x_n')$$
$$-\phi(y_1',y_2',\dots,y_m',x_{m+1}',\dots,x_n')\,\phi(y_1,y_2,\dots,y_m,x_{m+1},\dots,x_n)]\,|x_{m+1}\dots x_n\rangle\wedge|x_{m+1}'\dots x_n'\rangle\,d^{n-m}x'\,d^{n-m}x. \tag{9}$$

The Lagrange's identity takes the form

$$||\vec{a}||^2||\vec{b}||^2-|\vec{a}.\vec{b}|^2=||\vec{a}\wedge\vec{b}||^2.$$

By this identity, one may rewrite the entanglement measure $E_{\mathcal{M}}^2$ constructed as follows:

$$\left(\int_t^u|f(\vec{x})|^2d\vec{x}\right)\left(\int_t^u|g(\vec{x})|^2d\vec{x}\right)-\left|\int_t^u f(\vec{x})g^*(\vec{x})\right|^2=\iint_t^u|f(\vec{x})g(\vec{x}')-f(\vec{x}')g(\vec{x})|^2d\vec{x}'d\vec{x},$$

$$\mathcal{E}_{\mathcal{M}}^2=2\iint\Bigg[\left(\int|\phi(y_1'y_2'\dots y_m',x_{m+1},\dots,x_n)|^2d^{n-m}x\right)\left(\int|\phi(y_1y_2\dots y_m,x_{m+1},\dots,x_n)|^2d^{n-m}x\right)$$
$$-\left|\int\phi(y_1'y_2'\dots y_m',x_{m+1},\dots,x_n)\phi^*(y_1y_2\dots y_m,x_{m+1},\dots,x_n)d^{n-m}x\right|^2\Bigg]d^myd^my',$$

$$\mathcal{E}_{\mathcal{M}}^2=2\left[1-\iint\left|\int\phi(y_1'y_2'\dots y_m',x_{m+1},\dots,x_n)\phi^*(y_1y_2\dots y_m,x_{m+1},\dots,x_n)d^{n-m}x\right|^2d^myd^my'\right], \tag{10}$$

noting the normalization of ϕ. This may elegantly be written in terms of Φ and Λ_m as

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \mathrm{Re}\int \Phi(\mathrm{X})\Phi^*(\Lambda_m \mathrm{X}) d^{2n}\mathrm{X}\right]. \tag{11}$$

Hence, for maximal entanglement, one needs $\Phi(\mathrm{X})$ and $\Phi(\Lambda_m \mathrm{X})$ to be orthogonal, i.e., their inner product must vanish, so that $\mathcal{E}_{\mathcal{M}}^2$ takes the maximum value of 2. On the contrary, when $\Phi(\mathrm{X}) = \Phi(\Lambda_m \mathrm{X})$, their inner product takes the maximum overlap of 1, thereby implying separability with $\mathcal{E}_{\mathcal{M}}^2 = 0$. This is one of the important results of the paper on the geometry of entanglement in CV pure systems.

A. Gaussian CV states

We consider the example of a general pure Gaussian CV state to evaluate our criterion and provide the condition for separability, and we analyze the GEM for the case of a general two-mode Gaussian state:

$$\psi_1(x_1,\ldots,x_n) = \mathcal{N}_1 \exp\left(-\frac{1}{2}\left[\sum_{k=1}^{n} a_k x_k^2 + \sum_{k,j;j>k}^{n} c_{kj} x_k x_j\right]\right), \tag{12}$$

where $a_k \in \mathbb{R}$, $a_k \geqslant 0$, $c_{kj} \in \mathbb{C}$, and $\mathcal{N}_1$ is the appropriate normalization term for the wave function. Consider the separability of m degrees labeled by $x_1,\ldots,x_m$ from the n available degrees of freedom. For separability across the $m|(n-m)$ bipartition, one needs

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\left[\sum_{k=1}^{m} a_k y_k'^2 + \sum_{k=m+1}^{n} a_k x_k^2 + \sum_{k,j=1;j>k}^{k,j=m} c_{kj} y_k' y_j' + \sum_{k=1,j=m+1;j>k}^{k=m,j=n} c_{kj} y_k' x_j + \sum_{k,j=m+1;j>k}^{k,j=n} c_{kj} x_i x_j\right]\right)\\
&\quad\times \exp\left(-\frac{1}{2}\left[\sum_{k=1}^{m} a_k y_k^2 + \sum_{k=m+1}^{n} a_k x_k'^2 + \sum_{k,j=1;j>k}^{k,j=m} c_{kj} y_k y_j + \sum_{k=1,j=m+1}^{k=m,j=n} c_{kj} y_k x_j' + \sum_{k,j=m+1;j>k}^{k,j=n} c_{kj} x_i' x_j'\right]\right)\\
&= \exp\left(-\frac{1}{2}\left[\sum_{k=1}^{m} a_k y_k'^2 + \sum_{k=m+1}^{n} a_k x_k'^2 + \sum_{k,j=1;j>k}^{k,j=m} c_{kj} y_k' y_j' + \sum_{k=1,j=m+1}^{k=m,j=n} c_{kj} y_k' x_j' + \sum_{k,j=m+1;j>k}^{k,j=n} c_{kj} x_i' x_j'\right]\right)\\
&\quad\times \exp\left(-\frac{1}{2}\left[\sum_{k=1}^{m} a_k y_k^2 + \sum_{k=m+1}^{n} a_k x_k^2 + \sum_{k,j=1;j>k}^{k,j=m} c_{kj} y_k y_j + \sum_{k=1,j=m+1}^{k=m,j=n} c_{kj} y_k x_j + \sum_{k,j=m+1;j>k}^{k,j=n} c_{kj} x_i x_j\right]\right).
\end{aligned}$$

This simplifies to the requirement

$$\sum_{k=1,j=m+1}^{k=m,j=n} c_{kj}(y_k' x_j + y_k x_j') = \sum_{k=1,j=m+1}^{k=m,j=n} c_{kj}(y_k' x_j' + y_k x_j),$$

which can only be true for arbitrary values of y_i', x_i, x_i', and y_i, iff

$$\begin{aligned}
&c_{kj} = \ \forall\, k \in [1,m] \quad \text{and} \quad j \in [m+1,n], \ \text{or}\\
&\mathbb{V}_{k,j} = \mathbb{V}_{k,j}^T = 0\ \forall\, k \in [1,m] \quad \text{and} \quad j \in [m+1,n], \ \text{or}\\
&\mathbb{M}\mathbb{V}(1_{n\times n} - \mathbb{M}) = \mathbb{M}\mathbb{V}^T(1_{n\times n} - \mathbb{M}) = 0_{n\times n},
\end{aligned} \tag{13}$$

where $\mathbb{V} = \sum^{-1}$ is the inverse of the covariance matrix of the Gaussian. This is a necessary and sufficient condition for separability of the general Gaussian wave function $\psi_1(\vec{x})$ in n degrees of freedom across the bipartitions $m|(n-m)$. Moreover, one may say that the system is entangled across the bipartitions iff $\exists\, k, j$ such that $c_{kj} \neq 0$ for some $k \in [1,m]$ and $j \in [m+1,n]$.

To analyze the GEM for the case of a general two-mode Gaussian state,

$$\psi_2(x_1,x_2) = \mathcal{N}_2 e^{-\frac{1}{2}(ax_1^2 + bx_2^2 + cx_1x_2)}, \tag{14}$$

where $a, b \in \mathbb{R}$, $a, b > 0$, and c is either purely real or imaginary, and

$$\begin{aligned}
\mathcal{N}_2 &= \left[\frac{2\pi}{\sqrt{4ab - c^2}}\right]^{\frac{-1}{2}} \quad \text{if } c \text{ is real}\\
&= \left[\frac{\pi}{\sqrt{ab}}\right]^{\frac{-1}{2}} \quad \text{if } c = im, m \in \mathbb{R},
\end{aligned}$$

with $i \equiv \sqrt{-1}$. Using Eq. (12), one may compute the GEM across the modes as

$$\begin{aligned}
\mathcal{E}_{\mathcal{M}}^2 &= 2\left[1 - \frac{\sqrt{4ab - c^2}}{2\sqrt{ab}}\right] \quad \text{if c is real}\\
&= 2\left[1 - \frac{2\sqrt{ab}}{\sqrt{4ab + m^2}}\right] \quad \text{if } c = im, m \in \mathbb{R}.
\end{aligned}$$

Clearly, $E = 0$ iff $c = 0$ for both the cases. The entanglement depends on the parameter c. When c is real, one requires $-2\sqrt{ab} < c < 2\sqrt{ab}$, so that the state remains normalizable and hence physical. For this case, as $c \to \pm 2\sqrt{ab}$, $E^2 \to 2$. When c is purely imaginary with $c = im$, $m \in \mathbb{R}$, as $m \to \pm\infty$, $E^2 \to 2$ asymptotically.

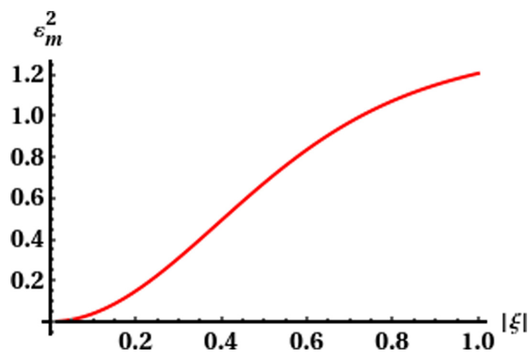

FIG. 1. Variation of the entanglement measure ($\mathcal{E}_m^2$) for a pair-coherent state with $|\xi|$. All the axes are dimensionless.

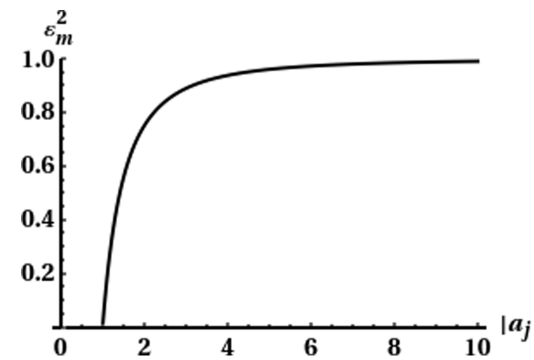

FIG. 2. Variation of the entanglement measure ($\mathcal{E}_m^2$) for a generalized superposition state with $|a_j|$. All the axes are dimensionless.

B. The pair-coherent state

The pair-coherent state [21] is given by

$$|\xi, 0\rangle = \frac{1}{\sqrt{I_0(2|\xi|)}} \sum \frac{\xi^i}{i!} |i, i\rangle, \tag{15}$$

where $I_0(2|\xi|)$ is the modified Bessel function of order zero. By using the GEM for pure states, for separability across the $m|(n-m)$ bipartition, one needs

$$\sum N_{ii} N'_{jj} |i, j\rangle\langle i, j| = \sum N_{j,i} N'_{j,i} |j, i\rangle\langle j, i|. \tag{16}$$

For $i \neq j$, by using Eq. (12), the entanglement measure is given by

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \frac{1}{I_0(2|\xi|)^2} \sum \frac{|\xi|^{4i}}{i!^4}\right]. \tag{17}$$

We plot the GEM for the pair-coherent state (15) with $|\xi|$ in Fig. 1. Clearly, for nonzero values of $|\xi|$, ($\mathcal{E}_m^2$) is nonzero. This implies entanglement in the pair-coherent state. For small values of $|\xi|$, ($\mathcal{E}_m^2$) increases slowly, and then it saturates at larger values.

C. Superposition of squeezed states

In Ref. [22], the first kind of superposition of the j squeezed vacuum state which has the same squeezing value in the Fock basis is given by

$$|\xi\rangle_j = N_j \sum (-e^{i\theta} \tanh r)^{jn} \frac{\sqrt{(2jn)!}}{2^{jn}(jn)!} |2jn\rangle, \tag{18}$$

where

$$N_j = \left(\sum \frac{(2jn)!}{2^{2jn}(jn)!} (\tanh r)^{2jn}\right)^{-\frac{1}{2}}.$$

By using Eq. (12), the entanglement measure is given by

$$\mathcal{E}_{\mathcal{M}}^2 = 0. \tag{19}$$

The generalized, second kind of superposition of a state with a different squeezing value and a differenct weight factor is

$$|\psi\rangle = \left(\sum_{p=1}^{l} \sum_{j=1}^{l} \frac{a_p a_j^*}{\sqrt{\cosh(q_p - q_j)}}\right)^{-\frac{1}{2}} \sum_{j=0}^{l-1} a_j |\xi_j\rangle, \tag{20}$$

where $\xi_j = q_j e^{i\theta_j}$, and a_j's are the weight factors. By using the GEM, the entanglement measure is

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \sum \frac{1}{|a_j|^2}\right]. \tag{21}$$

For superposition of the first kind, the entanglement measure is found to be 0, and from Fig. 2, the entanglement measure for the generalized second kind of superposition state increases slowly and then saturates when the weight factor reaches maximum.

D. Non-Gaussian CV Bell state

From Ref. [7], the non-Gaussian continuous variable state is expressed as

$$\psi_{\rm ng} = \sqrt{\frac{2}{\pi}} (p x_1 + q x_2) e^{-\frac{(x_1^2 + x_2^2)}{2}}. \tag{22}$$

The state is a composite system of bosonic particles formed from the ground and excited states of the harmonic oscillators [23]. The experimental scheme of this state has already been proposed [24]. The Peres-Horodecki criterion [25,26] is only sufficient for Eq. (22). Agarwal and Biswas [7] show inseparability of the state (22) via inequalities which are also applicable for the state (22).

By using the GEM, across the $m|(n-m)$ bipartitions, one finds

$$N_m N_m^* |m, n\rangle\langle m, n| + N_n N_n^* |n, m\rangle\langle n, m| + N_m^* N_n |m, m\rangle\langle n, n| + N_m N_n^* |n, n\rangle\langle m, m|.$$

For $m \neq n$, the measure of entanglement is given by

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \frac{2}{\pi} p^2 q^2\right]. \tag{23}$$

Clearly, we can see that the state is entangled and $\mathcal{E}_{\mathcal{M}}^2 = 0$ iff p or q is zero. The entanglement depends on the parameter p or q.

IV. GEM FOR MIXED CV STATES

We can now define the GEM for a general mixed continuous-variable systems. For N degree-of-freedom continuous-variable mixed states, the GEM is defined as

$$\mathcal{G}(\rho) = \min_{\{a_i, |\psi^i\rangle\}} \sum_i a_i E^2_{\mathcal{M}}(|\psi^i\rangle), \tag{24}$$

where ρ is any mixed state which is a convex combination of $\{a_i, |\psi^i\rangle\}$ of pure states,

$$\rho = \sum_i a_i |\psi^i\rangle\langle|\psi^i\rangle|.$$

To find the GEM of a mixed state, it is important to consider the nonuniqueness of the of the pure state decomposition. Here we consider convex hull construction [27], a general extension method to define the GEM for mixed continuous-variable states. We briefly discuss convex hull construction since we need this for one of the main results. Consider P to be a convex set and $Q \subset P$ to be an arbitrary subset. Let $F : Q \to \mathbb{R} \cup \{+\infty\}$. We then define the function $\text{coF} : P \to \mathbb{R} \cup \{+\infty\}$ by

$$\text{coF}(S) = \inf \left\{ \sum_i r_i F(x_i) | x_i \to Q, \sum_i r_i x_i = S \right\}, \tag{25}$$

where infimum is over all convex combinations with $\sum_i r_i \geqslant 0$, $\sum_i r_i = 1$, and infimum over an empty set is $+\infty$. Now consider an example: the entropy [19] is $E(a, b) = \text{tr}_a(lna - lb)$. In this notation, the definition of general entanglement or the relative entropy is

$$E_{\text{RE}}(\rho) = \text{CoE}(\rho).$$

We provide a general method for a class of continuous-variable mixed states via the above method which satisfies the following condition: An arbitrary state ρ is invariant under transformation such that $\rho = \rho'$.

ρ remains invariant under the transformation $\mathcal{X} \to \wedge_m \mathcal{X}$.

Proof. One can define

$$\tilde{\rho} = |\psi\rangle\langle\psi^*| = \int \phi(N_1\mathcal{X})\phi(N_2\mathcal{X}) N_1 |\mathcal{X}\rangle\langle\mathcal{X}| N_2^T d\mathcal{X}. \tag{26}$$

The matrix element of $\tilde{\rho}$ could be written as

$$\tilde{\rho} = \tilde{\Phi}(\mathcal{X}) N_1 |\mathcal{X}\rangle\langle\mathcal{X}| N_2^T.$$

Under the transpose, where the transposition is done on the $\tilde{\mathcal{M}}$ subsystem,

$$\tilde{\rho} \to \tilde{\Phi}(\mathcal{X}) N_1 \wedge_m |\mathcal{X}\rangle\langle\mathcal{X}| \wedge_m^T N_2^T = \tilde{\rho}_{(\wedge_m \mathcal{X})},$$

$$\tilde{\rho}' = \int \phi(N_1\mathcal{X})\phi(N_2\mathcal{X}) N_1 \wedge_m |\mathcal{X}\rangle\langle\mathcal{X}| \wedge_m^T N_2^T d\mathcal{X}. \tag{27}$$

Since the integration is on $\mathcal{X}$, $\tilde{\rho}'$ does not change under the substitution $\mathcal{X} \to \wedge_m \mathcal{X}$:

$$\tilde{\rho}' = \phi(N_1 \wedge_m \mathcal{X})\phi(N_2 \wedge_m \mathcal{X}) N_1 |\mathcal{X}\rangle\langle\mathcal{X}| N_2^T d\mathcal{X}.$$

Hence, $\rho = \rho'$.

In principle, one can have a set of states for which $\rho = \rho'$; then it is sufficient to perform the optimization over the set. If any mixed CV state satisfies the above, then this method can be successfully implemented to find the GEM for the state. Note that this method is directly connected to other methods that we discuss in Sec. III.

Now we show that the GEM is a "good" measure of entanglement [19] which satisfies all the following three conditions. The necessary conditions the measure of entanglement $\mathcal{G}(\rho)$ has to satisfy are as follows.

(i) $\mathcal{G}(\rho) = 0$ iff ψ is separable.

(ii) $\mathcal{G}(\rho)$ is invariant under local unitary operations.

(iii) The measure of entanglement cannot increase under local general measurements + classical communication.

To satisfy condition (i), it is sufficient to demand that $\mathcal{G}(\rho) = 0$, iff $\tilde{\Phi}(\mathcal{X}) = \tilde{\Phi}(\wedge_m \mathcal{X})$. Because of the invariance of ρ under $\mathcal{X} \to \wedge_m \mathcal{X}$, condition (ii) is automatically satisfied.

$\tilde{\Phi}(\mathcal{X})\tilde{\Phi}^*(\wedge_m \mathcal{X})$ is nonincreasing under every completely positive, trace-preserving map.

Proof. A complete measurement is given as a unitary operation + partial tracing on extended Hilbert space. For any completely positive, trace-preserving map σ, i.e., $\sigma(\mathcal{X}) = \sum W_i \mathcal{X} W_i^\dagger$ and $\sum_i W_i^\dagger W_i = 1$, where W is an operator satisfying the completeness relation $\sum_i W_i^\dagger W_i = 1$, G is any measure between two states λ and ω and is defined as $G(\lambda||\omega)$.

Vedral *et al.* [19] presented the following set of sufficient conditions.

(T1) Unitary operations leave $G(\lambda||\omega)$ invariant, i.e., $G(\lambda||\omega) = G(U\lambda U^\dagger||U\omega U^\dagger)$.

(T2) $G(\text{Tr}_\omega\lambda||\text{Tr}_\omega\omega) \leqslant \lambda||\omega$, where Tr_ρ is a partial trace.

(T3) $G(\lambda \otimes |\alpha\rangle\langle\alpha|||\omega \otimes |\alpha\rangle\langle\alpha|) = G(\lambda||\omega)$.

Let us define $V = \sum_i W_i \otimes |i\rangle\langle\eta|$, where $|i\rangle$ is an orthonormal basis and η is a unit vector, and

$$V^\dagger V = 1 \otimes |\eta\rangle\langle\eta|.$$

There is a unitary operator U such that

$$U(\mathcal{X} \otimes |\eta\rangle\langle\eta|)U^\dagger = \sum_{ij} W_i \mathcal{X} W_i^\dagger \otimes |i\rangle\langle j|,$$

$$\text{Tr}\{U(\mathcal{X} \otimes |\eta\rangle\langle\eta|)U^\dagger\} = \sum_i W_i \mathcal{X} W_i^\dagger. \tag{28}$$

Using condition (T2),

$$\phi[\text{Tr}_2\{U(\mathcal{X} \otimes |\eta\rangle\langle\eta|)U^\dagger\}]\phi^*[\text{Tr}_2\{U(\lambda_m\mathcal{X} \otimes |\eta\rangle\langle\eta|U^\dagger\}]$$
$$\leqslant \phi[U(\mathcal{X} \otimes |\eta\rangle\langle\eta|)U^\dagger]\phi^*[U(\lambda_m\mathcal{X} \otimes |\eta\rangle\langle\eta|)U^\dagger].$$

Using (T3),

$$\phi(\mathcal{X} \otimes |\eta\rangle\langle\eta|)\phi^*(\lambda_m\mathcal{X} \otimes |\eta\rangle\langle\eta|) = \phi(\mathcal{X})\phi^*(\lambda_m\mathcal{X}).$$

This proves the above condition.

Our measure has a statistical operational basis that might enable experimental determination of the quantitative degree of entanglement. Now the GEM for multiparty CV states is straightforward. We examine the GEM for a experimentally realized state in the following section.

Phase-matched squeezed state

The phase-randomized two-mode squeezed vacuum state [28] is given by

$$\rho_2 = \sum (1 - r) r^n |n\rangle\langle n| \otimes |n\rangle\langle n|, \tag{29}$$

where $r = \tanh \xi$ (ξ is a complex squeezing parameter). One can observe that Eq. (29) has no entanglement because it is a convex mixture of tensor product states $|n\rangle\langle n| \otimes |n\rangle\langle n|$ and the convex mixture is considered as a classical mixture of product states. In Eq. (29), the phase is equally distributed,

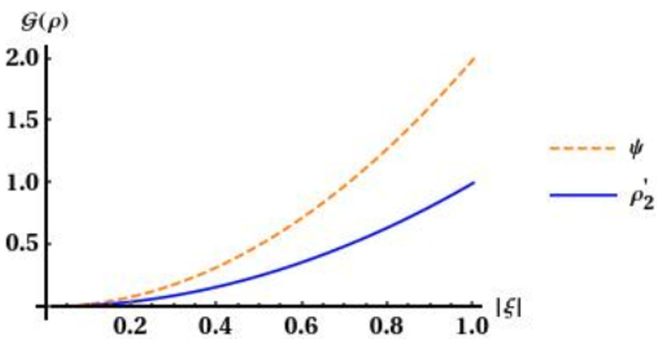

FIG. 3. Variation of the entanglement measure $\mathcal{G}(\rho)$ for a phase-matched squeezed state and a squeezed vacuum state with $|\xi|$. All the axes are dimensionless.

called the fully phase-randomized state. Consider the phase is not equally distributed, the phase-matched squeezed state is given by

$$\rho_2' = \sum_{m,n} p(m,n)(1-r)r^{\frac{m+n}{2}}|m,m\rangle\langle n,n|, \tag{30}$$

where $p(m,n) = \exp[\frac{-\sigma^2(m-n)^2}{2}]$, $r = \xi^2$, and σ is the variance. ρ_2' is experimentally accessible and the entanglement test can be performed experimentally. The GEM is given as

$$\mathcal{G}(\rho) = 2\left[1 - \sum_m (1-\xi^2)^2 \xi^{4m}\right]. \tag{31}$$

We show the variation of the GEM $\mathcal{G}(\rho)$ with $|\xi|$ in Fig. 3. In the case of the phase-matched squeezed state, when $|\xi| = 0$, $\mathcal{G}(\rho)$ starts increasing. Then, when $|\xi|$ increases slowly, it increases, and when $|\xi|$ tends to 1, it saturates. For better understanding of the entanglement of the phase-matched squeezed state, we have compared the GEMs of this state and of the squeezed vacuum states $\psi = \sqrt{1-|\xi|^2}\sum \xi^r|r,r\rangle$ in Fig. 3. Clearly, $\mathcal{G}(\rho)$ grows much faster, and varies linearly with $|\xi|$, than that of the phase-matched squeezed state, and at $|\xi| = 1$, the squeezed vacuum states attains maximum value. The entanglement measure $\mathcal{G}(\rho)$ of the squeezed vacuum state is larger than that of the phase-matched squeezed state.

V. DISCUSSION AND CONCLUSION

One may note the dynamics in the phase space by considering the Wigner transform on both the sides of $\tilde{\Phi}(\mathcal{X}) = \tilde{\Phi}(\wedge_m\mathcal{X})$, where $\mathcal{X}_{2n\times 1} \equiv [\begin{smallmatrix}\vec{x}_1\\ \vec{x}_2\end{smallmatrix}]$ with $\vec{x}_1$ and $\vec{x}_2$ being some general coordinates of the system. Noting that $\vec{x}_1$ and $\vec{x}_2$ are independent, the Wigner transform of the left-hand side is

$$\tilde{W}(\mathcal{X},\mathcal{P}) = \left(\frac{1}{\pi}\right)^{2n}\int e^{2i\mathcal{P}\cdot\mathcal{Y}}\tilde{\Phi}(\mathcal{X}-\mathcal{Y})\tilde{\Phi}^*(\mathcal{X}+\mathcal{Y})d\mathcal{Y}$$
$$= W(\vec{x}_1,\vec{p}_1)W(\vec{x}_2,\vec{p}_2), \tag{32}$$

where $\mathcal{Y} \equiv [\begin{smallmatrix}\vec{y}_1\\ \vec{y}_2\end{smallmatrix}]$, $\mathcal{P} \equiv [\begin{smallmatrix}\vec{p}_1\\ \vec{p}_2\end{smallmatrix}]$, and W is the corresponding Wigner function of the given state $|\psi\rangle$. Under separability, $\tilde{W}(\mathcal{X},\mathcal{P})$ may equivalently be written as

$$= \left(\frac{1}{\pi}\right)^{2n}\int e^{2i\mathcal{P}\cdot\mathcal{Y}}\tilde{\Phi}[\wedge_m(\mathcal{X}-\mathcal{Y})]\tilde{\Phi}^*[\wedge_m(\mathcal{X}+\mathcal{Y})]d\mathcal{Y},$$

and since the integration runs on $\mathcal{Y}$, transforming $\mathcal{Y} \to \wedge_m\mathcal{Y}$ does not change the integral. Therefore, under separability,

$$\tilde{W}(\mathcal{X},\mathcal{P}) = \left(\frac{1}{\pi}\right)^{2n}\int e^{2i\mathcal{P}\cdot\wedge_m\mathcal{Y}}\tilde{\Phi}(\wedge_m\mathcal{X}-\mathcal{Y})$$
$$\times\,\tilde{\Phi}^*(\wedge_m\mathcal{X}+\mathcal{Y})d\mathcal{Y}$$
$$= \tilde{W}(\wedge_m\mathcal{X},\wedge_m\mathcal{P}),$$

noting that $\mathcal{P}\cdot\wedge_m\mathcal{Y} = \wedge_m\mathcal{P}\cdot\mathcal{Y}$, $d\mathcal{Y} = d(\wedge_m\mathcal{Y})$, and $\wedge_m^2 = 1$. Therefore, $\tilde{W}(\xi)$ being invariant under the coordinate transformation $\xi \to (\begin{smallmatrix}\wedge_m & 0\\ 0 & \wedge_m\end{smallmatrix})\xi$, where $\xi_{4n\times 1} = [\begin{smallmatrix}\mathcal{X}\\ \mathcal{P}\end{smallmatrix}]$, is a necessary and sufficient condition for separability. Considering $\mathrm{Tr}(\rho_{\mathrm{PT}}^4) = \mathrm{Tr}[(\rho^{\mathcal{M}})^2 \otimes (\rho^{\tilde{\mathcal{M}}})^2] = (\mathrm{Tr}[(\rho^{\mathcal{M}})^2])^2$, one may rewrite entanglement measure as

$$\mathcal{E}_{\mathcal{M}}^2 = 2\left[1 - \sqrt{\mathrm{Tr}\left(\rho_{\mathrm{PT}}^4\right)}\,\right]$$
$$= 2\left[1 - \sqrt{\int W_{\mathrm{PT}}^4(\vec{x},\vec{p})d\vec{x}d\vec{p}}\,\right]. \tag{33}$$

For the case of 2 degrees of freedom, therefore, if $\int W_{\mathrm{PT}}^4(x_1,p_1,x_2,p_2)d\vec{x}d\vec{p} = 1$ for a given pure state, one may write $W_{\mathrm{PT}}(x_1,p_1,x_2,p_2) = W(x_1,p_1,x_2,-p_2)$ as shown by Simon [6] under separability. Observing that $\mathrm{Tr}(\rho_{\mathrm{PT}}) = 1$ and $\mathrm{Tr}(\rho_{\mathrm{PT}}^2) = 1$ for any given pure density matrix ρ, one may note that, when ρ_{PT} is positive semidefinite, the eigenvalues must be either 0 or 1 with a multiplicity of 1 in the DV case. So any higher powers of ρ_{PT} would also have unit trace. Hence, ρ is separable iff ρ_{PT} is positive-semidefinite.

We conclude by commenting on the connections to other widely used measures in the literature to show their equivalence to the GEM. The Hilbert-Schmidt distance D between two density matrices, ρ_1 and ρ_2, is $D_{\rho_1}^2(\rho_2) \equiv ||\rho_1 - \rho_2||_{\mathrm{HS}}^2 = \mathrm{Tr}[(\rho_1-\rho_2)^2]$ has been widely used to study the geometry and structure of entanglement with connections to negativity and positive partially transposed states [17,29]. Now using Eqs. (3) and (4), we get

$$\tilde{\rho} - \tilde{\rho_{\mathrm{PT}}} = \int[\phi(\mathcal{N}_1\mathcal{X})\phi(\mathcal{N}_2\mathcal{X}) - \phi(\mathcal{N}_1\wedge_m\mathcal{X})$$
$$\times\,\phi(\mathcal{N}_2\wedge_m\mathcal{X})]\mathcal{N}_1|\mathcal{X}\rangle\langle\mathcal{X}|\mathcal{N}_2^T\,d\mathcal{X}. \tag{34}$$

It is evident that the GEM can be instead interpreted as

$$\mathcal{E}_{\mathcal{M}}^2 = ||\tilde{\rho}_1 - \tilde{\rho}_2||_{\mathrm{HS}}^2. \tag{35}$$

This is shown below to be related to the Hilbert-Schmidt distance of the reduced density matrix from the maximally mixed state using Lagrange's identity. Consider the case of the general DV system with the density matrix $\rho_{N\times N}$. Taking $\rho_1 = \frac{1}{N}1_{N\times N}$, one may write the distance of ρ to the maximally mixed state ρ_1 as

$$D^2(\rho) = \mathrm{Tr}\left(\frac{1}{N^2}1 + \rho^2 - \frac{2}{N}\rho\right) = \mathrm{Tr}(\rho^2) - \frac{1}{N},$$

noting $\mathrm{Tr}(\rho) = 1$ and $\mathrm{Tr}(1) = N$. In the CV case as $N \to \infty$, $D^2(\rho) = \mathrm{Tr}(\rho^2) = 1 - E^2(\rho)/2$, where E is the generalized entanglement measure. One may conversely use this property to geometrically define define a maximally mixed CV states, noting from Eq. (13) that one can have the following identity for CV states,

$$||\tilde{\rho}_1 - \tilde{\rho}_2||_{\mathrm{HS}}^2 + 2||\rho_{\mathcal{M}} - \rho_1||_{\mathrm{HS}}^2 = 2.$$

On the same note, one can show the equivalence of the von Neumann entropy as an entanglement measure to the GEM. The entropy of a density matrix ρ is defined as $S = -\mathrm{Tr}(\rho \ln \rho) = \langle \ln \rho \rangle$, where $\langle \cdot \rangle$ denotes the expectation value. Expanding S around a pure state $\rho^2 = \rho$, that is, the nonnegative matrix $1 - \rho$, and noting that $E^2/2 = 1 - \mathrm{Tr}(\rho^2) = \mathrm{Tr}[\rho(1-\rho)] = \langle 1 - \rho \rangle$, one infers

$$\begin{aligned} S &= -\langle \ln \rho \rangle \\ &= \langle 1 - \rho \rangle + \langle (1-\rho)^2/2 \rangle + \langle (1-\rho)^3/3 \rangle + \cdots, \end{aligned}$$

and therefore, $S = \frac{E^2}{2} +$ residual. Clearly, iff $E = 0$, the residual term vanishes, giving $S = 0$; otherwise when $E > 0$, the residual remains positive, giving $S > E^2/2$ for any state ρ. Therefore, S and E are equivalent in characterizing separable states and entanglement among entangled states faithfully. It is, however, faster computationally to calculate the GEM than it is to find the von Neumann entropy, as it does not require diagonalization of the density matrix.

The convex roof construction involves optimization and is usually hard. The entanglement of an arbitrary mixed continuous-variable state is not a simple task. In this paper, we defined entanglement for pure continuous-variable states and extended the concept to mixed continuous-variable states via the convex roof construction. We evaluated the measure for several classes of continuous-variable states. However, it is not clear whether the same method is useful for the mixture of states which have white or colored noise. The persistence of subplanck structure in mixed continuous-variable states is made possible with specific environmental conditions [30]. We hope our work provides insights into the geometry and structure of entanglement in general in both pure and mixed continuous-variable systems by proving a family of faithful entanglement measures and equivalent forms of necessary and sufficient conditions for separability across arbitrary bipartitions. We believe the results hold deep connections to the recent works on the nature of quantum correlations in many-body systems [31], monogamy of entanglement [32,33], and fundamental aspects of quantum mechanics, including the uncertainty principle and commutation relations [6,8,10].

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APPENDIX: PROOF OF LAGRANGE'S IDENTITY

Considering the right-hand side of Eq. (11),

$$\begin{aligned}
&= \iint_{\vec{t}}^{\vec{u}} |f(\vec{x})g(\vec{x}') - f(\vec{x}')g(\vec{x})|^2 d\vec{x}' d\vec{x} \\
&= \frac{1}{2} \iint_{\vec{t}}^{\vec{u}} |f(\vec{x})g(\vec{x}') - f(\vec{x}')g(\vec{x})|^2 d\vec{x}' d\vec{x} \\
&= \frac{1}{2} \iint_{\vec{t}}^{\vec{u}} [f(\vec{x})g(\vec{x}') - f(\vec{x}')g(\vec{x})] \\
&\quad [f^*(\vec{x})g^*(\vec{x}') - f^*(\vec{x}')g^*(\vec{x})] d\vec{x}' d\vec{x} \\
&= \frac{1}{2} \iint_{\vec{t}}^{\vec{u}} [|f(\vec{x})|^2 |g(\vec{x}')|^2 - 2\mathrm{Re}[f(\vec{x})g(\vec{x}')f^*(\vec{x}')g^*(\vec{x})] \\
&\quad + |f(\vec{x}')|^2 |g(\vec{x})|^2] d\vec{x}' d\vec{x} \\
&= \left(\int_{\vec{t}}^{\vec{u}} |f(\vec{x})|^2 d\vec{x} \right) \left(\int_{\vec{t}}^{\vec{u}} |g(\vec{x}')|^2 d\vec{x} \right) \\
&\quad - \mathrm{Re} \iint_{\vec{t}}^{\vec{u}} f(\vec{x})g(\vec{x}')f^*(\vec{x}')g^*(\vec{x}) d\vec{x}' d\vec{x} \\
&= \left(\int_{\vec{t}}^{\vec{u}} |f(\vec{x})|^2 d\vec{x} \right) \left(\int_{\vec{t}}^{\vec{u}} |g(\vec{x}')|^2 d\vec{x} \right) - \left| \int_{\vec{t}}^{\vec{u}} f(\vec{x})g^*(\vec{x}) d\vec{x} \right|^2 \\
&= \text{Left-hand side,}
\end{aligned}$$

hence the identity.

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